

MIXED NORM SPACES AND REARRANGEMENT INVARIANT ESTIMATES

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ABSTRACT. Our main goal in this work is to further improve the mixed norm estimates due to Fournier [13], and also Algervik and Kolyada [1], to more general rearrangement invariant (r.i.) spaces. In particular we find the optimal domains and the optimal ranges for these embeddings between mixed norm spaces and r.i. spaces.

1. INTRODUCTION

Estimates on mixed norm spaces $\mathcal{R}(X, Y)$ (see Definition 3.2) already appeared in the works of Gagliardo [14] and Nirenberg [24], to prove an endpoint case of the classical Sobolev embeddings. However, a more systematic approach to these spaces was first shown explicitly by Fournier [13].

Recall that the Sobolev space $W^1 L^p(I^n)$, $1 \leq p < \infty$, consists of all functions in $L^p(I^n)$ whose first-order distributional derivatives also belong to $L^p(I^n)$. We write $W_0^1 L^p(I^n)$ for the closure of smooth function, with compact support, in $W^1 L^p(I^n)$.

The classical Sobolev embedding theorem claims that

$$(1) \quad W_0^1 L^p(I^n) \hookrightarrow L^{pn/(n-p)}(I^n), \quad 1 \leq p < n.$$

Sobolev [27] proved this embedding for $p > 1$, but his method, based on integral representations, did not work when $p = 1$. That case was settled affirmatively by Gagliardo [14] and Nirenberg [24], who first observed that

$$(2) \quad W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty),$$

and then, using an iterated form of Hölder's inequality, completed the proof; i.e.,

$$W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(I^n).$$

Later, a new approach based on properties of mixed norm spaces was introduced by Fournier [13] and was subsequently developed, via different methods, by various authors, including Blei [7], Milman [23], Algervik and Kolyada [1] and Kolyada [19, 20]. To be more precise, the central part of Fournier's work was to prove

$$(3) \quad \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n),$$

and then taking into account (2), he obtained the following improvement of (1):

$$(4) \quad W_0^1 L^1(I^n) \hookrightarrow L^{n',1}(I^n).$$

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The embedding (4) is due to Peetre [25], and it can be also traced in the work of Kerman and Pick [18], where a characterization of Sobolev embeddings in rearrangement invariant (r.i.) spaces was obtained.

Since the embedding (3) was first proved, many other proofs and different extensions have appeared in the literature. In particular, relations between mixed norm spaces of Lorentz spaces were studied in [1], where it was shown, for instance,

$$(5) \quad \mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)',1}, L^1), \quad n \geq 2.$$

All these works provide us a strong motivation to better understand the embeddings between mixed norm spaces, as well as to provide a characterization of (3) for more general r.i. spaces.

The paper is organized as follows. A precise definition of r.i. spaces, and also other definitions and properties concerning function spaces to be used throughout the paper can be found in Section 2.

In Section 3 we introduce the Benedek-Panzone spaces and the mixed norm spaces. Here, among other things, we find an explicit formula for the Peetre K -functional for the couple of mixed norm spaces $(\mathcal{R}(X, L^\infty), L^\infty)$.

In Section 4, we explore connections between mixed norm spaces. It is important to observe that, for any $k \in \{1, \dots, n\}$,

$$(6) \quad \mathcal{R}_k(X_1, Y_1) \hookrightarrow \mathcal{R}_k(X_2, Y_2) \Leftrightarrow \begin{cases} X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}), \\ Y_1(I) \hookrightarrow Y_2(I), \end{cases}$$

(see Lemma 4.1). That is, looking at each specific component, the embeddings are trivial. However, there are examples, for instance (5), showing that if in (6) we replace Benedek-Panzone spaces by the global mixed norm spaces, then the corresponding equivalence is not longer true. This fact illustrates that mixed norm spaces may have a much more complicated structure than Benedek-Panzone spaces. Therefore, it is natural to analyze embeddings between mixed norm spaces.

Motivated by this problem, in Section 4, we find necessary and sufficient conditions for the existence of the following types of embeddings:

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2), \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty), \quad \mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y).$$

After this discussion, in Section 4 our analysis focuses on a particular embedding:

$$(7) \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^{p,1}), \quad 1 \leq p < \infty.$$

To be more specific, Theorem 4.12 provides a characterization of the smallest mixed norm space of the form $\mathcal{R}(Y, L^{p,1})$ in (7), once the mixed norm space $\mathcal{R}(X, L^\infty)$ is given. In order to prove it, we develop a method which can be used to obtain most of the known results on embeddings of mixed norm spaces due to Algervik and Kolyada [1].

Section 5 is devoted to the study of the embedding (3) for more general r.i. spaces. In particular, Theorem 5.2 gives necessary and sufficient conditions for the following embedding to hold:

$$(8) \quad \mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

A general consequence of Theorem 5.2 is contained in Theorem 5.4, which provides a characterization of the largest space of the form $\mathcal{R}(X, L^\infty)$ in (8), once the r.i.

space $Z(I^n)$ is given. Finally, for a fixed mixed norm space $\mathcal{R}(X, L^\infty)$, Theorem 5.7 describes the smallest r.i. space for which (8) holds.

Some remarks about the notation: The measure of the unit ball in \mathbb{R}^n will be represented by ω_n . As usual, we use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C , independent of all important parameters, such that $A \leq CB$. The equivalence $A \approx B$ means that $A \lesssim B$ and $B \gtrsim A$. Finally, the arrow \hookrightarrow stands for a continuous embedding.

2. PRELIMINARIES

We collect in this section some basic notations and concepts that will be useful in what follows.

Let $n \in \mathbb{N}$, with $n \geq 1$ and let $I \subset \mathbb{R}$ be a finite interval. We write $\mathcal{M}(I^n)$ for the set of all real-valued measurable functions on I^n .

Given $f \in \mathcal{M}(I^n)$, its distribution function λ_f is defined by

$$\lambda_f(t) = |\{x \in I^n : |f(x)| > t\}|, \quad t \geq 0,$$

(where $|\cdot|$ denotes the Lebesgue measure) and the decreasing rearrangement f^* of f is given by

$$f^*(t) = \inf\{s \geq 0 : \lambda_f(s) \leq t\}, \quad t \geq 0.$$

It is easily seen that if g is any radial function on \mathbb{R}^n of the form $g(x) = f^*(\omega_n|x|^n)$, for some $f \in \mathcal{M}(I^n)$, then $g^* = f^*$.

As usual, we shall use the notation $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$.

A basic property of rearrangements is the Hardy-Littlewood inequality (cf. e.g. [5, Theorem II.2.2]), which says:

$$\int_{I^n} |f(x)g(x)| dx \leq \int_0^{|I|^n} f^*(t)g^*(t) dt, \quad f, g \in \mathcal{M}(I^n).$$

A rearrangement invariant Banach function space $X(I^n)$ (briefly an r.i. space) is the collection of all $f \in \mathcal{M}(I^n)$ for which $\|f\|_{X(I^n)} < \infty$, where $\|\cdot\|_{X(I^n)}$ satisfies the following properties:

- (A1) $\|\cdot\|_{X(I^n)}$ is a norm;
- (A2) if $0 \leq g \leq f$ a.e., then $\|g\|_{X(I^n)} \leq \|f\|_{X(I^n)}$;
- (A3) if $0 \leq f_j \uparrow f$ a.e., then $\|f_j\|_{X(I^n)} \uparrow \|f\|_{X(I^n)}$;
- (A4) $\|\chi_{I^n}\|_{X(I^n)} < \infty$;
- (A5) $\int_{I^n} |f(x)| dx \lesssim \|f\|_{X(I^n)}$;
- (A6) if $f^* = g^*$, then $\|f\|_{X(I^n)} = \|g\|_{X(I^n)}$.

Given an r.i. space $X(I^n)$, the set

$$X'(I^n) = \left\{ f \in \mathcal{M}(I^n) : \int_{I^n} |f(x)g(x)| dx < \infty, \text{ for any } g \in X(I^n) \right\},$$

equipped with the norm

$$\|f\|_{X'(I^n)} = \sup_{\|g\|_{X(I^n)} \leq 1} \int_{I^n} |f(x)g(x)| dx$$

is called the associate space of $X(I^n)$. It turns out that $X'(I^n)$ is again an r.i. space [5, Theorem I.2.2]. The fundamental function of an r.i. space $X(I^n)$ is given by

$$\varphi_X(t) = \|\chi_E\|_{X(I^n)},$$

where $|E| = t$ and χ_E denotes the characteristic function of the set $E \subset I^n$. It is known [26, Theorem 5.2] that if $X(I^n)$ is an r.i. space, then

$$(9) \quad X(I^n) \neq L^\infty(I^n) \Leftrightarrow \lim_{t \rightarrow 0^+} \varphi_X(t) = 0.$$

The Lorentz space Λ_{φ_X} consists of all $f \in \mathcal{M}(I^n)$ for which the expression

$$\|f\|_{\Lambda_{\varphi_X}} = \|f\|_{L^\infty(I^n)} \varphi_X(0^+) + \int_0^{|I|^n} f^*(t) \varphi'_X(t) dt$$

is finite. It is well-known [5, Theorem II.5.13] that if $X(I^n)$ is an r.i. space then,

$$\Lambda_{\varphi_X} \hookrightarrow X(I^n).$$

A basic tool for working with r.i. spaces is the Hardy-Littlewood-Pólya Principle which asserts that if $f \in X(I^n)$ and

$$\int_0^t g^*(s) ds \leq \int_0^t f^*(s) ds, \quad 0 < t < |I|^n,$$

then $g \in X(I^n)$ and $\|g\|_{X(I^n)} \leq \|f\|_{X(I^n)}$.

For later purposes, let us recall the Luxemburg *Representation Theorem* [5, Theorem II.4.10]. It says that given an r.i. space $X(I^n)$, there exists another r.i. space $\overline{X}(0, |I|^n)$ such that

$$f \in X(I^n) \iff f^* \in \overline{X}(0, |I|^n),$$

and in this case $\|f\|_{X(I^n)} = \|f^*\|_{\overline{X}(0, |I|^n)}$.

Next, we define the Boyd indices of an r.i. space. First we introduce the dilation operator:

$$E_t f(s) = \begin{cases} f(s/t), & \text{if } 0 \leq s \leq \min(|I|^n, t|I|^n), \\ 0, & \text{otherwise,} \end{cases} \quad t > 0, \quad f \in \mathcal{M}(0, |I|^n).$$

Let us recall that the operator E_t is bounded on $\overline{X}(0, |I|^n)$, for every r.i. space $X(I^n)$ and for every $t > 0$.

By means of the norm of E_t on $\overline{X}(0, |I|^n)$, denoted by $h_X(t)$, we define the lower and upper Boyd indices of $X(I^n)$ as

$$\underline{\alpha}_X = \sup_{0 < t < 1} \frac{\log h_X(t)}{\log(t)} \quad \text{and} \quad \overline{\alpha}_X = \inf_{1 < t < \infty} \frac{\log h_X(t)}{\log(t)}.$$

It is easy to see that $0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq 1$.

Basic examples of r.i. spaces are the Lebesgue spaces $L^p(I^n)$, $1 \leq p \leq \infty$. Another important class of r.i. spaces is provided by the Lorentz spaces. We recall that the Lorentz space $L^{p,q}(I^n)$, with $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = \infty$, is the r.i. space consisting of all $f \in \mathcal{M}(I^n)$ for which the quantity

$$\|f\|_{L^{p,q}(I^n)} = \begin{cases} \left(\int_0^{|I|^n} [s^{1/p} f^*(s)]^q \frac{ds}{s} \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{0 < t < |I|^n} t^{1/p} f^*(t), & \text{if } q = \infty \end{cases}$$

is finite. Observe that $L^{p,p}(I^n) = L^p(I^n)$. For a comprehensive treatment of r.i. spaces we refer the reader to [5].

Finally, let us recall some special results from Interpolation Theory, which we shall need in what follows. For further information on this topic see [5, 6, 10].

Given a pair of compatible Banach spaces (X_0, X_1) (compatible in the sense that they are continuously embedded into a common Hausdorff topological vector space), their K -functional is defined, for each $f \in X_0 + X_1$, by

$$K(f, t; X_0, X_1) := \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \quad t > 0.$$

The fundamental result concerning the K -functional is:

Theorem 2.1. *Let (X_0, X_1) and (Y_0, Y_1) be two compatible pairs of Banach spaces and let T be a sublinear operator satisfying*

$$T : X_0 \rightarrow Y_0, \quad \text{and} \quad T : X_1 \rightarrow Y_1.$$

Then, there exists a constant $C > 0$ (depending only on the norms of T between X_0 and Y_0 and between X_1 and Y_1) such that

$$K(Tf, t; Y_0, Y_1) \leq CK(f, Ct; X_0, X_1), \quad \text{for every } f \in X_0 + X_1 \text{ and } t > 0.$$

The K -functional for pairs of Lorentz spaces $L^{p,q}(I^n)$ is given, up to equivalence, by the following result.

Theorem 2.2. *(Holmstedt's formulas [17, Theorem 4.2]) Let $p_0 = q_0 = 1$ or $1 < p_0 < \infty$ and $1 \leq q_0 < \infty$. Let $1/\alpha = 1/p_0 - 1/p_1$. Then,*

$$K(f, t; L^{p_0, q_0}(I^n), L^\infty(I^n)) \approx \left(\int_0^{tp_0} [s^{1/p_0 - 1/q_0} f^*(s)] ds \right)^{1/q_0}, \quad \text{for } t > 0.$$

3. MIXED NORM SPACES

In what follows and throughout the paper we shall assume $n \geq 2$.

Our goal in this section is to present some basic properties of mixed norm spaces. Let $k \in \{1, \dots, n\}$. We write \hat{x}_k for the point in I^{n-1} obtained from a given vector $x \in I^n$ by removing its k th coordinate. That is,

$$\hat{x}_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in I^{n-1}.$$

Moreover, for any $f \in \mathcal{M}(I^n)$, we use the notation $f_{\hat{x}_k}$ for the function obtained from f , with \hat{x}_k fixed. Let us recall that, since f is measurable, $f_{\hat{x}_k}$ is also measurable.

For later purposes, let us first enumerate some geometric properties of the projections. We refer to the book [16] for basic facts on this topic.

Let $E \subset I^n$ be any measurable set and let $\hat{x}_k \in I^{n-1}$ be fixed. The x_k -section of E is defined as

$$E(\hat{x}_k) = \{x_k \in I : (\hat{x}_k, x_k) \in E\}.$$

Let us emphasize that since E is measurable, its x_k -section is also measurable. The essential projection of E onto the hyperplane $x_k = 0$ is defined as

$$\Pi_k^* E = \{\hat{x}_k \in I^{n-1} : |E(\hat{x}_k)| > 0\}.$$

An important result is the so-called Loomis-Whitney inequality [21, Theorem 1] which says

$$(10) \quad |E| \leq \prod_{k=1}^n |\Pi_k E|^{1/(n-1)},$$

where $\Pi_k E$ is the orthogonal projection of E into the coordinate hyperplane $x_k = 0$.

An improvement of (10) was given in [13, 1], using the measures of the essential projections, was proved:

$$(11) \quad |E| \leq \prod_{k=1}^n |\Pi_k^* E|^{1/(n-1)}.$$

We now recall the Benedek-Panzone spaces, which were introduced in [4]. For further information on this topic see [11, 8, 9, 3].

Definition 3.1. Let $k \in \{1, \dots, n\}$. Given two r.i. spaces $X(I^{n-1})$ and $Y(I)$, the Benedek-Panzone space $\mathcal{R}_k(X, Y)$ is defined as the collection of all $f \in \mathcal{M}(I^n)$ satisfying

$$\|f\|_{\mathcal{R}_k(X, Y)} = \|\psi_k(f, Y)\|_{X(I^{n-1})} < \infty,$$

where $\psi_k(f, Y)(\hat{x}_k) = \|f(\hat{x}_k, \cdot)\|_{Y(I)}$.

Buhvalov [11] and Blozinski [8] proved that $\mathcal{R}_k(X, Y)$ is a Banach function space. Moreover Boccuto, Buhvalov, and Sambucini [9] proved that $\mathcal{R}_k(X, Y)$ is an r.i. space, if and only if $X = Y = L^p$.

Now, we shall give the definition of the mixed norm spaces, sometimes also called symmetric mixed norm spaces.

Definition 3.2. Given two r.i. spaces $X(I^{n-1})$ and $Y(I)$, the mixed norm space $\mathcal{R}(X, Y)$ is defined as

$$\mathcal{R}(X, Y) = \bigcap_{k=1}^n \mathcal{R}_k(X, Y).$$

For each $f \in \mathcal{R}(X, Y)$, we set $\|f\|_{\mathcal{R}(X, Y)} = \sum_{k=1}^n \|f\|_{\mathcal{R}_k(X, Y)}$.

It is not difficult to verify that $\mathcal{R}(X, Y)$ is a Banach function space.

Since the pioneering works of Gagliardo [14], Nirenberg [24], and Fournier [13], many useful properties and generalizations of these spaces have been studied, via different methods, by various authors, including Blei [7], Milman [23], Algervik and Kolyada [1], and Kolyada [19, 20].

All these works, together with the embedding (2), provide us a strong motivation to better understand the mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$. For this, we start with a useful lemma:

Lemma 3.3. Let $f \in \mathcal{M}(I^n)$ and let $E_\alpha = \{x \in I^n : |f(x)| > \alpha\}$, with $\alpha \geq 0$. Then,

$$\Pi_k^* E_\alpha = \{\hat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\hat{x}_k) > \alpha\}.$$

Proof. To prove this lemma, it is enough to consider $\alpha > 0$. Let us see that

$$(12) \quad \{\hat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\hat{x}_k) > \alpha\} \subset \Pi_k^* E_\alpha.$$

In fact, if $\widehat{x}_k \notin \Pi_k^* E_\alpha$, then, by definition of $\Pi_k^* E_\alpha$, we have

$$|\{x_k \in I : |f(\widehat{x}_k, x_k)| > \alpha\}| = 0.$$

But

$$\psi_k(f, L^\infty)(\widehat{x}_k) = \inf \{s \geq 0 : |\{x_k \in I : |f(\widehat{x}_k, x_k)| > s\}| = 0\}, \quad \widehat{x}_k \in I^{n-1},$$

and hence, we get $\psi_k(f, L^\infty)(\widehat{x}_k) \leq \alpha$. As a consequence, we have

$$\widehat{x}_k \notin \{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha\}.$$

This proves that (12) holds. To complete the proof, it only remains to see that

$$\Pi_k^* E_\alpha \subset \{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha\}.$$

In fact, if $\widehat{x}_k \in \Pi_k^* E_\alpha$, then $|\{x_k \in I : |f(\widehat{x}_k, x_k)| > \alpha\}| > 0$. So, by definition, $\psi_k(f, L^\infty)(\widehat{x}_k) > \alpha$. Therefore, $\widehat{x}_k \in \{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha\}$. Thus, the proof is complete. \square

As an immediate consequence of inequality (11) and Lemma 3.3, we have the following inequality:

Corollary 3.4. *Let $f \in \mathcal{M}(I^n)$. Then, for any $t > 0$,*

$$\lambda_f(t) \leq \left(\prod_{k=1}^n \lambda_{\psi_k(f, L^\infty)}(t) \right)^{1/(n-1)}.$$

Our next goal is to describe the K -functional for pairs of the form $(\mathcal{R}(X, L^\infty), L^\infty)$. Then, we shall apply it to the characterization of the r.i. hull of a mixed norm space of the form $\mathcal{R}(X, L^\infty)$ (see Theorem 5.7). We refer to the paper [23], for other results concerning interpolation of mixed norm spaces.

We first prove a lower bound for the K -functional for the couple of mixed norm spaces $(\mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y))$.

Lemma 3.5. *Let $X(I^{n-1})$ and $Y(I)$ be r.i. spaces. Then,*

$$\sum_{k=1}^n \|\psi_k^*(f, Y)\chi_{(0,t)}\|_{\overline{X}(0,|I|^{n-1})} \lesssim K(f, \varphi_X(t), \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)), \quad 0 < t < |I|^{n-1}.$$

Proof. We fix $0 < t < |I|^{n-1}$ and $k \in \{1, \dots, n\}$. If $f = f_0 + f_1$, with $f_0 \in \mathcal{R}(X, Y)$ and $f_1 \in \mathcal{R}(L^\infty, Y)$, then

$$\psi_k(f, Y)(\widehat{x}_k) \leq \psi_k(f_0, Y)(\widehat{x}_k) + \psi_k(f_1, Y)(\widehat{x}_k), \quad \widehat{x}_k \in I^{n-1}.$$

So, it holds that

$$\psi_k^*(f, Y)(t) \leq \psi_k^*(f_0, Y)(t) + \psi_k^*(f_1, Y)(0) = \psi_k^*(f_0, Y)(t) + \|f_1\|_{\mathcal{R}_k(L^\infty, Y)}.$$

Therefore, we have

$$\begin{aligned} \|\psi_k^*(f, Y)\chi_{(0,t)}\|_{\overline{X}(0,|I|^{n-1})} &\leq \|\psi_k^*(f_0, Y)\chi_{(0,t)}\|_{\overline{X}(0,|I|^{n-1})} + \varphi_X(t) \|f_1\|_{\mathcal{R}_k(L^\infty, Y)} \\ &\leq \|f_0\|_{\mathcal{R}(X, Y)} + \varphi_X(t) \|f_1\|_{\mathcal{R}(L^\infty, Y)}. \end{aligned}$$

Hence, taking the infimum over all decompositions of f of the form $f = f_0 + f_1$, with $f_0 \in \mathcal{R}(X, Y)$ and $f_1 \in \mathcal{R}(L^\infty, Y)$, we obtain

$$\|\psi_k^*(f, L^\infty)\chi_{(0,t)}\|_{\overline{X}(0,|I|^{n-1})} \leq K(f, \varphi_X(t), \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)),$$

for any $k \in \{1, \dots, n\}$ and $0 < t < |I|^{n-1}$. Consequently,

$$\sum_{k=1}^n \left\| \psi_k^*(f, Y) \chi_{(0,t)} \right\|_{\overline{X}(0, |I|^{n-1})} \leq nK(f, \varphi_X(t), \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)), \quad 0 < t < |I|^{n-1},$$

from which the result follows. \square

Theorem 3.6. *Let $X(I^{n-1})$ be an r.i. space and let $f \in \mathcal{R}(X, L^\infty) + L^\infty(I^n)$. Then,*

$$K(f, \varphi_X(t), \mathcal{R}(X, L^\infty), L^\infty) \approx \sum_{k=1}^n \left\| \psi_k^*(f, L^\infty) \chi_{(0,t)} \right\|_{\overline{X}(0, |I|^{n-1})}, \quad 0 < t < |I|^{n-1},$$

where $\varphi_X(t)$ is the fundamental function of $X(I^{n-1})$.

Proof. In view of Lemma 3.5, we only need to prove

$$K(f, \varphi_X(t), \mathcal{R}(X, L^\infty), L^\infty) \lesssim \sum_{k=1}^n \left\| \psi_k^*(f, L^\infty) \chi_{(0,t)} \right\|_{\overline{X}(0, |I|^{n-1})}, \quad 0 < t < |I|^{n-1}.$$

For this, we fix any $0 < t < |I|^{n-1}$. Then, we define

$$\alpha_t = \sum_{j=1}^n \psi_j^*(f, L^\infty)(t),$$

$$F(x) = \begin{cases} f(x) - \frac{\alpha_t f(x)}{|f(x)|}, & \text{if } x \in A_t = \{x \in I^n : |f(x)| > \alpha_t\}, \\ 0, & \text{otherwise,} \end{cases}$$

and $G = f - F$. Let $k \in \{1, \dots, n\}$ be fixed. Then, for any $\widehat{x}_k \in I^{n-1}$,

$$F_{\widehat{x}_k}(y) = \begin{cases} f_{\widehat{x}_k}(y) - \frac{\alpha_t f(\widehat{x}_k)(y)}{|f_{\widehat{x}_k}(y)|}, & \text{if } y \in A_t(\widehat{x}_k), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$A_t(\widehat{x}_k) = \{y \in I : (\widehat{x}_k, y) \in A_t\} = \{y \in I : |f_{\widehat{x}_k}(y)| > \alpha_t\}.$$

Thus, for any $s \geq 0$ and $\widehat{x}_k \in I^{n-1}$,

$$\begin{aligned} \lambda_{F_{\widehat{x}_k}}(s) &= |\{y \in I : |F_{\widehat{x}_k}(y)| > s\}| = |\{y \in A_t(\widehat{x}_k) : ||f_{\widehat{x}_k}(y)| - \alpha_t| > s\}| \\ &= |\{y \in I : |f_{\widehat{x}_k}(y)| - \alpha_t > s\}| = \lambda_{f_{\widehat{x}_k}}(s + \alpha_t). \end{aligned}$$

Now, let us suppose that $\widehat{x}_k \notin \Pi_k^* A_t$. Then, by definition, $\lambda_{f_{\widehat{x}_k}}(\alpha_t) = 0$. As a consequence, we have that if $\widehat{x}_k \notin \Pi_k^* A_t$, then $\lambda_{F_{\widehat{x}_k}}(s) = 0$, for any $s \geq 0$. Therefore, for any $s \geq 0$, it holds that

$$\lambda_{F_{\widehat{x}_k}}(s) = \begin{cases} \lambda_{f_{\widehat{x}_k}}(s + \alpha_t), & \text{if } \widehat{x}_k \in \Pi_k^* A_t, \\ 0, & \text{otherwise.} \end{cases}$$

So, Lemma 3.3 implies that, for any $s \geq 0$,

$$\lambda_{F_{\widehat{x}_k}}(s) = \begin{cases} \lambda_{f_{\widehat{x}_k}}(s + \alpha_t), & \text{if } \widehat{x}_k \in \{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha_t\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for any $\widehat{x}_k \in \{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha_t\}$, we get

$$\begin{aligned} \psi_k(f, L^\infty)(\widehat{x}_k) &= \inf \{y > 0 : \lambda_{F_{x_k}}(y) = 0\} = \inf \{y > 0 : \lambda_{f_{x_k}}(y + \alpha_t) = 0\} \\ &= \psi_k(f, L^\infty)(\widehat{x}_k) - \alpha_t. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|F\|_{\mathcal{R}_k(X, L^\infty)} &= \|(\psi_k(f, L^\infty)(\widehat{x}_k) - \alpha_t) \chi_{\{\widehat{x}_k \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > \alpha_t\}}(\widehat{x}_k)\|_{X(I^{n-1})} \\ &= \|(\psi_k^*(f, L^\infty) - \alpha_t) \chi_{(0, \lambda_{\psi_k(f, L^\infty)}(\alpha_t))}\|_{\overline{X}(0, |I|^{n-1})}. \end{aligned}$$

But, by hypothesis, $\alpha_t = \sum_{j=1}^n \psi_j^*(f, L^\infty)(t)$ and hence we get

$$\|F\|_{\mathcal{R}_k(X, L^\infty)} \leq \|\psi_k^*(f, L^\infty) \chi_{(0, t)}\|_{\overline{X}(0, |I|^{n-1})}, \quad \text{for any } k \in \{1, \dots, n\}.$$

Therefore, using the above inequality, we obtain

$$\begin{aligned} K(f, \varphi_X(t), \mathcal{R}(X, L^\infty), L^\infty) &\leq \|F\|_{\mathcal{R}(X, L^\infty)} + \varphi_X(t) \|G\|_{L^\infty(\mathbb{R}^n)} \\ &= \sum_{k=1}^n \|F\|_{\mathcal{R}_k(L^1, L^\infty)} + \varphi_X(t) \alpha_t \\ &\leq \sum_{k=1}^n \|\psi_k^*(f, L^\infty) \chi_{(0, t)}\|_{\overline{X}(0, |I|^{n-1})} + \varphi_X(t) \alpha_t. \end{aligned}$$

But, it holds that

$$\varphi_X(t) \alpha_t \leq \sum_{k=1}^n \|\psi_k^*(f, L^\infty) \chi_{(0, t)}\|_{\overline{X}(0, |I|^{n-1})},$$

and hence, we have

$$K(f, \varphi_X(t), \mathcal{R}(X, L^\infty), L^\infty) \leq 2 \sum_{k=1}^n \|\psi_k^*(f, L^\infty) \chi_{(0, t)}\|_{\overline{X}(0, |I|^{n-1})}, \quad t > 0.$$

Thus, the proof is complete. \square

As a consequence we obtain the following result. Recall that $(A, B)_{\theta, q}$ stands for the real interpolation space of the couple (A, B) [5, Definition V.1.7]:

Corollary 3.7. *Let $X(I^{n-1})$ be an r.i. space. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then,*

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q} = \mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty),$$

with equivalent norms.

To prove it, we first need to recall a result concerning the K -functional of pairs of r.i. spaces. For further information see [22, 2].

Theorem 3.8. *Let $X(I^n)$ be an r.i. space. Then,*

$$K(f, \varphi_X(t), X, L^\infty) \approx \|f^* \chi_{(0, t)}\|_{\overline{X}(0, |I|^n)}, \quad t > 0.$$

Proof of Corollary 3.7. For the sake of simplicity, we prove this result only when $1 \leq q < \infty$. Let $f \in (\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}$. Then, by a change of variables, we get

$$\begin{aligned} \|f\|_{(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}}^q &= \int_0^\infty t^{-\theta q - 1} [K(f, t, \mathcal{R}(X, L^\infty), L^\infty)]^q dt \\ &= \int_0^\infty (\varphi_X(s))^{-\theta q - 1} [K(f, \varphi_X(s), \mathcal{R}(X, L^\infty), L^\infty)]^q d\varphi_X(s). \end{aligned}$$

Hence, using Theorem 3.6, we obtain

$$\begin{aligned} \|f\|_{(\mathcal{R}(L^1, L^\infty), L^\infty)_{\theta, q}}^q &\approx \int_0^\infty (\varphi_X(s))^{-\theta q - 1} \left[\sum_{k=1}^n \|\psi_k^*(f, L^\infty) \chi_{(0, s)}\|_{\overline{X}(0, |I|^{n-1})} \right]^q ds \\ &\geq \int_0^\infty (\varphi_X(s))^{-\theta q - 1} \|\psi_k^*(f, L^\infty) \chi_{(0, s)}\|_{\overline{X}(0, |I|^{n-1})}^q ds, \end{aligned}$$

for any $k \in \{1, \dots, n\}$. So, Theorem 3.8 implies that

$$\begin{aligned} \|f\|_{(\mathcal{R}(L^1, L^\infty), L^\infty)_{\theta, q}}^q &\gtrsim \int_0^\infty (\varphi_X(s))^{-\theta q - 1} [K(\psi_k(f, L^\infty), \varphi_X(s), X, L^\infty)]^q d\varphi_X(s) \\ &= \|\psi_k(f, L^\infty)\|_{(X, L^\infty)_{\theta, q}}^q = \|f\|_{\mathcal{R}_k((X, L^\infty)_{\theta, q}, L^\infty)}. \end{aligned}$$

As a consequence, we get

$$\|f\|_{\mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty)} = \sum_{k=1}^n \|f\|_{\mathcal{R}_k((X, L^\infty)_{\theta, q}, L^\infty)} \lesssim \|f\|_{(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}}.$$

Thus, we have seen that the embedding

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q} \hookrightarrow \mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty)$$

holds. Hence, to complete the proof, it only remains to see that

$$\mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty) \hookrightarrow (\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q},$$

also holds. To do it, we fix any $f \in \mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty)$. Then, using Theorem 3.6 and the subadditive property of $\|\cdot\|_{(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}}$, we get

$$\|f\|_{(\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}} \leq \sum_{k=1}^n \left(\int_0^\infty (\varphi_X(s))^{-\theta q - 1} \|\psi_k^*(f, L^\infty) \chi_{(0, s)}\|_{\overline{X}(0, |I|^{n-1})}^q d\varphi_X(s) \right)^{1/q}.$$

So, using Theorem 3.8, we obtain

$$\|f\|_{(\mathcal{R}(L^1, L^\infty), L^\infty)_{\theta, q}} \lesssim \sum_{k=1}^n \|\psi_k(f, L^\infty)\|_{(X, L^\infty)_{\theta, q}} = \|f\|_{\mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty)}.$$

That is, $\mathcal{R}((X, L^\infty)_{\theta, q}, L^\infty) \hookrightarrow (\mathcal{R}(X, L^\infty), L^\infty)_{\theta, q}$. Thus, the proof is complete. \square

4. EMBEDDINGS BETWEEN MIXED NORM SPACES

Our aim in this section is to characterize certain embeddings between mixed norm spaces. Before that, let us emphasize that relations between mixed norm spaces of Lorentz spaces were studied in [1], where it was shown, for instance,

$$(13) \quad \mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)', 1}, L^1), \quad n \geq 2.$$

Let us start with some preliminary lemmas:

Lemma 4.1. *Let $k \in \{1, \dots, n\}$. Let $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$, and $Y_2(I)$ be r.i. spaces. Then,*

$$\mathcal{R}_k(X_1, Y_1) \hookrightarrow \mathcal{R}_k(X_2, Y_2) \Leftrightarrow \begin{cases} X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}), \\ Y_1(I) \hookrightarrow Y_2(I). \end{cases}$$

Proof. To prove the implication “ \Rightarrow ”, we just have to apply the hypothesis to the functions

$$g_1(x) = f_1(\widehat{x}_k)\chi_{I^n}(x) \quad \text{and} \quad g_2(x) = f_2(x_k)\chi_{I^n}(x),$$

with $f_1 \in X_1(I^{n-1})$ and $f_2 \in Y_1(I)$. The converse follows from Definition 3.1. \square

It is important to observe that there are examples, for instance (13), showing that if in Lemma 4.1 we replace the Benedek-Panzone spaces by mixed norm spaces, then the corresponding equivalence is not longer true. However, we always have this result:

Lemma 4.2. *Let $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$, and $Y_2(I)$ be r.i. spaces. Then,*

$$X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}) \text{ and } Y_1(I) \hookrightarrow Y_2(I) \Rightarrow \mathcal{R}(X_1, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2).$$

Proof. It is immediate from Definition 3.2 and Lemma 4.1. \square

Lemma 4.1 and Lemma 4.2 show that it is natural to study when embeddings between mixed norm spaces are true. Motivated by this problem, we shall find necessary and sufficient conditions in the following cases:

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2), \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty), \quad \mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y).$$

Theorem 4.3. *For any r.i. spaces $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$ and $Y_2(I)$, if the following embedding*

$$\mathcal{R}(X_1, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2)$$

holds, then $Y_1(I) \hookrightarrow Y_2(I)$.

Proof. By Lemma 4.2, we may assume, without loss of generality, that the following embedding

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(L^1, Y_2)$$

holds. Also, we shall suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$. Let $r \in \mathbb{R}$ such that $0 < r < \min(a, b)$. Given any function $g \in \overline{Y}_1(I)$, with $\lambda_g(0) \leq 2r/n$, we define

$$f(x) = g^*\left(2\left|\sum_{i=1}^n x_i\right|\right)\chi_{(-r, r)^n}(x).$$

For any $k \in \{1, \dots, n\}$, we denote

$$\beta_k = \sum_{i=1, i \neq k}^n x_i, \quad \text{whenever } \widehat{x}_k \in (-r, r)^{n-1}.$$

Now, if $s \geq 0$, we have

$$\begin{aligned} \lambda_{f_{\widehat{x}_k}}(s) &= |\{x \in (-r, r) : g^*(2|x + \beta_k|) > s\}| \\ &= |\{x \in (-r, r) \cap (-r/n - \beta_k, r/n - \beta_k) : g^*(2|x + \beta_k|) > s\}| \\ &\leq |\{x \in (-r/n - \beta_k, r/n - \beta_k) : g^*(2|x + \beta_k|) > s\}| \\ &= |\{x \in (-r/n, r/n) : g^*(2|x|) > s\}| = \lambda_g(s). \end{aligned}$$

Thus,

$$\{s \geq 0 : \lambda_g(s) \leq t\} \subseteq \{s \geq 0 : \lambda_{f_{\widehat{x}_k}}(s) \leq t\}, \quad \text{for any } t \geq 0,$$

and so $f_{\widehat{x}_k}^* \leq g^*$. Hence, we get

$$\psi_k(f, Y_1)(\widehat{x}_k) \leq \|g^*\|_{\overline{Y}_1(0, |I|)}, \quad \widehat{x}_k \in (-r, r)^{n-1}.$$

Therefore,

$$\|f\|_{\mathcal{R}_k(L^\infty, Y_1)} \leq \|g^*\|_{\overline{Y}_1(0, |I|)}, \quad k \in \{1, \dots, n\}.$$

Hence, our assumption on g ensures that $f \in \mathcal{R}(L^\infty, Y_1)$ and

$$(14) \quad \|f\|_{\mathcal{R}(L^\infty, Y_1)} \leq n \|g^*\|_{\overline{Y}_1(0, |I|)}.$$

Thus, using $\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(L^1, Y_2)$ and (14), we get

$$(15) \quad \|f\|_{\mathcal{R}(L^1, Y_2)} \lesssim \|g^*\|_{\overline{Y}_1(0, |I|)}.$$

Now, let us compute $\|f\|_{\mathcal{R}(L^1, Y_2)}$. In order to do it, we fix any $k \in \{1, \dots, n\}$ and $\widehat{x}_k \in (0, r/n)^{n-1}$, and set

$$\gamma_k = \sum_{i=1, i \neq k}^n x_i.$$

As before, if $s \geq 0$, we have

$$\lambda_{f_{\widehat{x}_k}}(s) = |\{x \in (-r, r) \cap (-r/n - \gamma_k, r/n - \gamma_k) : g^*(2|x + \gamma_k|) > s\}|.$$

But, $0 < \gamma_k < r/n'$, so we obtain

$$\lambda_{f_{\widehat{x}_k}}(s) = |\{x \in (-r/n - \gamma_k, r/n - \gamma_k) : g^*(2|x + \gamma_k|) > s\}| = \lambda_g(s),$$

for any $s \geq 0$. As a consequence, if $\widehat{x}_k \in (0, r/n)^{n-1}$, then $f_{\widehat{x}_k}^* = g^*$. Thus,

$$\begin{aligned} \psi_k(f, Y_2)(\widehat{x}_k) &= \|f(\widehat{x}_k, \cdot)\|_{Y_2(I)} \chi_{(-r, r)^{n-1}}(\widehat{x}_k) \geq \|f(\widehat{x}_k, \cdot)\|_{Y_2(I)} \chi_{(0, r/n)^{n-1}}(\widehat{x}_k) \\ &= \|g^*\|_{\overline{Y}_2(0, |I|)} \chi_{(0, r/n)^{n-1}}(\widehat{x}_k), \end{aligned}$$

and so

$$\|f\|_{\mathcal{R}(L^1, Y_2)} \gtrsim \|g^*\|_{\overline{Y}_2(0, |I|)}.$$

Therefore, inequality (15) gives us that

$$(16) \quad \|g^*\|_{\overline{Y}_2(0, |I|)} \lesssim \|g^*\|_{\overline{Y}_1(0, |I|)},$$

for any $g \in Y(I)$, with $\lambda_g(0) \leq 2r/n$. Now, let us consider any $g \in Y_1(I)$. We define

$$g_1(x) = \max(|g(x)| - g^*(2r/n), 0) \operatorname{sgn} g(x),$$

and

$$g_2(x) = \min(|g(x)|, g^*(2r/n)) \operatorname{sgn} g(x).$$

Since $\lambda_{g_1}(0) \leq 2r/n$, the inequality (16), with g replaced by g_1 , implies that

$$(17) \quad \|g_1\|_{Y_2(I)} \lesssim \|g_1\|_{Y_1(I)}.$$

Thus, combining $g_1 \leq g$ a.e. with (17), we get

$$(18) \quad \|g_1\|_{Y_2(I)} \lesssim \|g\|_{Y_1(I)}.$$

On the other hand, by Hölder's inequality, we obtain

$$(19) \quad \|g_2\|_{Y_2(I)} \leq \varphi_{Y_2}(|I|) g^{**}(2r/n) \lesssim \|g\|_{Y_1(I)}.$$

Finally, using (18) and (19), we get

$$\|g\|_{Y_2(I)} = \|g_1 + g_2\|_{Y_2(I)} \lesssim \|g\|_{Y_1(I)}, \quad f \in Y_1(I),$$

and the proof is complete. \square

As a consequence we have the following corollary:

Corollary 4.4. *Let $X_2(I^{n-1})$, $Y_1(I)$ and $Y_2(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2) \Leftrightarrow Y_1(I) \hookrightarrow Y_2(I).$$

Proof. It follows from Lemma 4.2 and Theorem 4.3. \square

Another consequence of Theorem 4.3 is the following result:

Corollary 4.5. *Let $1 < p_1, p_3 < \infty$, $1 \leq q_1, q_3 \leq \infty$ and either $p_2 = q_2 = 1$, $p_2 = q_2 = \infty$ or $1 < p_2 < \infty$ and $1 \leq q_2 \leq \infty$. Then,*

$$\mathcal{R}(L^\infty, L^{p_1, q_1}) \hookrightarrow \mathcal{R}(L^{p_2, q_2}, L^{p_3, q_3}) \Leftrightarrow \begin{cases} p_3 < p_1, 1 \leq q_1, q_3 \leq \infty, \\ p_1 = p_3, 1 \leq q_1 \leq q_3 \leq \infty. \end{cases}$$

Proof. It follows from Theorem 4.3 and the classical embeddings for Lorentz spaces (see [5]). \square

Let us now study embeddings between mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$.

Theorem 4.6. *Let $X_1(I^{n-1})$ and $X_2(I^{n-1})$ be r.i. spaces. Then,*

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty) \Leftrightarrow X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}).$$

Proof. In view of Lemma 4.2, we only need to prove that if the embedding

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty)$$

holds, then $X_1(I^{n-1}) \hookrightarrow X_2(I^n)$. As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any $f \in X_1(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1} r^{n-1}$, we define

$$g(x) = \begin{cases} f^*(\omega_{n-1}|x|^{n-1}), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

We fix any $k \in \{1, \dots, n\}$. Then, it holds that

$$\psi_k(g, L^\infty)(\hat{x}_k) = \|g(\hat{x}_k, \cdot)\|_{L^\infty(I)} = f^*(\omega_{n-1}|\hat{x}_k|^{n-1}), \quad \text{if } \hat{x}_k \in B_{n-1}(0, r),$$

and $\psi_k(g, L^\infty)(\hat{x}_k) = 0$ otherwise. So, for any $k \in \{1, \dots, n\}$, we have

$$\|g\|_{\mathcal{R}_k(X_1, L^\infty)} = \|\psi_k(g, L^\infty)\|_{X_1(I^{n-1})} = \|f\|_{X_1(I^{n-1})}.$$

Hence, since we are assuming that $f \in X_1(I^{n-1})$, we obtain $g \in \mathcal{R}(X_1, L^\infty)$ and

$$\|g\|_{\mathcal{R}(X_1, L^\infty)} = n \|f\|_{X_1(I^{n-1})}.$$

So, using $\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty)$ and the previous inequality, we get

$$\|g\|_{\mathcal{R}(X_2, L^\infty)} \lesssim \|f\|_{X_1(I^{n-1})}.$$

But, as before,

$$\|g\|_{\mathcal{R}(X_2, L^\infty)} = n \|f\|_{X_2(I^{n-1})},$$

hence, we have

$$\|f\|_{X_2(I^{n-1})} \lesssim \|f\|_{X_1(I^{n-1})}.$$

This proves that if $f \in X_1(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$, then $f \in X_2(I^{n-1})$. The rest of the proof is essentially the same as in Theorem 4.3. \square

For a general r.i. space $Y(I)$, we have a similar result assuming some conditions on $X_1(I^{n-1})$.

Theorem 4.7. *Let $X_1(I^{n-1})$ be an r.i. space, with $\underline{\alpha}_{X_1} > 0$, and let $X_2(I^{n-1})$ and $Y(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y) \Leftrightarrow X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}).$$

Proof. As before, according to Lemma 4.2, it suffices to prove the necessary part of this result. Also, by Theorem 4.6, we assume that $Y(I) \neq L^\infty(I)$. Let us suppose that the embedding

$$\mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y)$$

holds and suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any function $f \in X_1(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$, we define

$$g(x) = \begin{cases} \int_{\omega_{n-1}|x|^{n-1}}^{\omega_{n-1}r^{n-1}} \frac{f^*(t)}{t\varphi_Y(2(t/\omega_{n-1})^{1/(n-1)})} dt, & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

We fix $k \in \{1, \dots, n\}$ and $\widehat{x}_k \in B_{n-1}(0, r)$. Using now (9), we obtain

$$\begin{aligned} \psi_k(g, Y)(\widehat{x}_k) &\leq \psi_k(g, \Lambda_{\varphi_Y})(\widehat{x}_k) = \psi_k(g, L^\infty)(\widehat{x}_k)\varphi_Y(0^+) + \int_0^{|\widehat{x}_k|} g_{\widehat{x}_k}^*(t)\varphi_Y'(t)dt \\ &= \int_0^{2((\lambda_f(0)/\omega_{n-1})^{1/(n-1)} - |\widehat{x}_k|)} \varphi_Y'(t)dt \\ &\quad \times \left(\int_{\omega_{n-1}(t/2 + |\widehat{x}_k|)^{n-1}}^{\omega_{n-1}r^{n-1}} \frac{f^*(s)}{s\varphi_Y(2(s/\omega_{n-1})^{1/(n-1)})} ds \right). \end{aligned}$$

Then, Fubini's theorem gives

$$\begin{aligned} \psi_k(g, Y)(\widehat{x}_k) &\lesssim \int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^{\lambda_f(0)} \frac{f^*(s)}{s\varphi_Y(2(s/\omega_{n-1})^{1/(n-1)})} \left(\int_0^{2(s/\omega_{n-1})^{1/(n-1)} - 2|\widehat{x}_k|} \varphi_Y'(t)dt \right) ds \\ &= \int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^{\lambda_f(0)} \frac{f^*(s)\varphi_Y(2(s/\omega_{n-1})^{1/(n-1)} - 2|\widehat{x}_k|)}{s\varphi_Y(2(s/\omega_{n-1})^{1/(n-1)})} ds. \end{aligned}$$

Hence, using that φ_Y is an increasing function, we deduce that

$$\psi_k(g, Y)(\widehat{x}_k) \lesssim \int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^{\lambda_f(0)} f^*(s) \frac{ds}{s}.$$

Since $\underline{\alpha}_{X_1} > 0$, [5, Theorem V.5.15] ensures that the integral operator

$$\int_t^{|\widehat{x}_k|^{n-1}} f^*(s) \frac{ds}{s},$$

is bounded on $\overline{X}_1(I^{n-1})$ and, as a consequence, we obtain

$$\|g\|_{\mathcal{R}_k(X_1, Y)} = \|\psi_k(g, Y)\|_{X_1(I^{n-1})} \lesssim \left\| \int_t^{\lambda_f(0)} f^*(s) \frac{ds}{s} \right\|_{\overline{X}_1(0, |I|^{n-1})} \lesssim \|f\|_{X_1(I^{n-1})},$$

for any $k \in \{1, \dots, n\}$. Hence, our assumption on f gives that $g \in \mathcal{R}(X_1, Y)$ and

$$(20) \quad \|g\|_{\mathcal{R}(X_1, Y)} \lesssim \|f\|_{X_1(I^{n-1})}.$$

So, using $\mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y)$ and (20), we get

$$(21) \quad \|g\|_{\mathcal{R}(X_2, Y)} \lesssim \|f\|_{X_1(I^{n-1})}.$$

We next find a lower estimate for $\|g\|_{\mathcal{R}(X_2, Y)}$. In fact, we fix $k \in \{1, \dots, n\}$ and $\widehat{x}_k \in B_{n-1}(0, r/2)$. Then, by Hölder's inequality, we get

$$(22) \quad \frac{1}{\varphi_{Y'}(2|\widehat{x}_k|)} \int_{B_1(0, |\widehat{x}_k|)} g(\widehat{x}_k, x_k) dx_k \leq \psi_k(g, Y)(\widehat{x}_k).$$

On the other hand, by a change of variables, it holds that

$$\begin{aligned} \int_{B_1(0, |\widehat{x}_k|)} g(\widehat{x}_k, x_k) dx_k &\approx \int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}} t^{1/(n-1)} \frac{dt}{t} \\ &\quad \times \left(\int_t^{\omega_{n-1}r^{n-1}} \frac{f^*(s)}{s\varphi_Y(2(s/\omega_{n-1})^{1/(n-1)})} ds \right), \end{aligned}$$

and so Fubini's theorem gives

$$\begin{aligned} \int_{B_1(0, |\widehat{x}_k|)} g(\widehat{x}_k, x_k) &\gtrsim \int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}} \frac{f^*(t)}{t\varphi_Y(2(t/\omega_{n-1})^{1/(n-1)})} dt \\ &\quad \times \left(\int_{\omega_{n-1}|\widehat{x}_k|^{n-1}}^t s^{1/(n-1)-1} ds \right) \\ &\gtrsim \int_{(3/2)^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}} \frac{f^*(t)(t^{1/(n-1)} - \omega_{n-1}^{1/(n-1)}|\widehat{x}_k|)}{t\varphi_Y(2(t/\omega_{n-1})^{1/(n-1)})} dt \\ &\gtrsim \varphi_{Y'}(2|\widehat{x}_k|) f^*(2^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}). \end{aligned}$$

Hence, using (22), we obtain

$$f^*(2^{n-1}\omega_{n-1}|\widehat{x}_k|^{n-1}) \lesssim \psi_k(g, Y)(\widehat{x}_k), \quad \widehat{x}_k \in B_{n-1}(0, r/2).$$

Therefore,

$$\|f^*\|_{\overline{X}_2(0, |I|^{n-1})} \lesssim \|\chi_{(0, \lambda_f(0)/2)} f^*(2t)\|_{\overline{X}_2(0, |I|^{n-1})} \lesssim \|\psi_k(g, Y)\|_{X_2(I^{n-1})}.$$

Thus, using (21), we get

$$\|f\|_{X_2(I^{n-1})} \lesssim \|f\|_{X_1(I^{n-1})}.$$

This proves that if $f \in X_1(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$, then $f \in X_2(I^{n-1})$. The general case can be treated as at the end of the proof of Theorem 4.3. \square

As a consequence of Theorem 4.6 and Theorem 4.7, we obtain the following result:

Corollary 4.8. *Let $1 < p_1, p_3 < \infty$, $1 \leq q_1, q_3 \leq \infty$ and either $p_2 = q_2 = 1$, $p_2 = q_2 = \infty$ or $1 < p_2 < \infty$ and $1 \leq q_2 \leq \infty$. Then*

$$\mathcal{R}(L^{p_1, q_1}, L^{p_2, q_2}) \hookrightarrow \mathcal{R}(L^{p_3, q_3}, L^{p_2, q_2}) \Leftrightarrow \begin{cases} p_3 < p_1, 1 \leq q_1, q_3 \leq \infty, \\ p_1 = p_3, 1 \leq q_1 \leq q_3 \leq \infty. \end{cases}$$

Proof. It follows from Theorem 4.6, Theorem 4.7 and the classical embeddings for Lorentz spaces (see [5]). \square

Finally, let us study the embedding

$$(23) \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^{p,1}).$$

Let us start by analyzing the case $p = 1$, with $n = 2$. The following result will be useful for our purposes.

Lemma 4.9. *Let $X(I)$ be an r.i. space. Then,*

$$\mathcal{R}(L^1, X) \hookrightarrow \mathcal{R}(X, L^1).$$

Proof. Let $f \in \mathcal{R}(L^1, X)$ and $k \in \{1, 2\}$. Then, using Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned} \|f\|_{\mathcal{R}_k(X, L^1)} &= \sup_{\|g\|_{X'(I)} \leq 1} \int_I \int_I |g(\hat{x}_k) f(\hat{x}_k, x_k)| d\hat{x}_k dx_k \\ &\leq \int_I \psi_k(f, X)(\hat{x}_k) d\hat{x}_k = \|f\|_{\mathcal{R}_k(L^1, X)} \leq \|f\|_{\mathcal{R}(L^1, X)}. \end{aligned}$$

That is, $\mathcal{R}(L^1, X) \hookrightarrow \mathcal{R}(X, L^1)$ and the proof is complete. \square

Corollary 4.10. *For any couple of r.i. spaces $X_1(I)$ and $X_2(I)$, we have*

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^1).$$

Proof. Using Lemma 4.2 and Lemma 4.9, we get

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^1, X_2) \hookrightarrow \mathcal{R}(X_2, L^1),$$

as we wanted to see. \square

Now, let us consider the embedding (23), for the cases $n = 2$ and $1 < p < \infty$ or $n \geq 3$ and $1 \leq p < \infty$. In particular, we shall provide a characterization of the smallest mixed norm space of the form $\mathcal{R}(Y, L^{p,1})$ in (23) once the mixed norm space $\mathcal{R}(X_1, L^\infty)$ is given. In order to do it, we begin with a preliminary lemma.

Lemma 4.11. *Let $X(I^{n-1})$ be an r.i. space. Then, the functional defined by*

$$(24) \quad \|f\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})} = \|f^*(t)t^{-1/p(n-1)}\|_{\overline{X}(0, |I|^{n-1})}, \quad f \in \mathcal{M}_+(I^{n-1}),$$

is an r.i. norm.

Proof. The positivity and homogeneity of $\|\cdot\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})}$ are clear. Next, let f and g be measurable functions on I^n . Then,

$$\begin{aligned} \|f + g\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})} &= \|t^{-1/p(n-1)}(f + g)^*(t)\|_{\overline{X}(0, |I|^{n-1})} \\ &= \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^{|I|^{n-1}} (f + g)^*(t)t^{-1/p(n-1)} h^*(t) dt. \end{aligned}$$

Since $(f + g)^* \prec f^* + g^*$ (cf. [5, Theorem II.3.4]), Hardy's lemma (cf. [5, Proposition II.3.6]) implies that

$$\begin{aligned} \|f + g\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})} &\lesssim \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^{|I|^{n-1}} f^*(t) t^{-1/p(n-1)} h^*(t) dt \\ &\quad + \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^{|I|^n} g^*(t) t^{-1/p(n-1)} h^*(t) dt. \end{aligned}$$

Using Hölder's inequality, we get the subadditivity property of $\|\cdot\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})}$. The proof of (A2)-(A4) and (A6) for $\|\cdot\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})}$ requires only the corresponding axioms for $\|\cdot\|_{\overline{X}(0, |I|^{n-1})}$, hence we shall omit them. Finally, to prove property (A5), we fix any $f \in \mathcal{M}(I^{n-1})$. Then,

$$\begin{aligned} \|t^{-1/p(n-1)} f^*(t)\|_{\overline{X}(0, |I|^{n-1})} &\gtrsim \int_0^{I^{n-1}} t^{-1/p(n-1)} f^*(t) dt \\ &\geq \int_0^{I^{n-1}} t^{-1/p(n-1)} f^*(|I|^{1/p} t^{-1/(p(n-1))+1}) dt. \end{aligned}$$

Thus, by a change of variables, we obtain,

$$\|t^{-1/p(n-1)} f^*(t)\|_{\overline{X}(0, |I|^{n-1})} \gtrsim \int_0^{I^{n-1}} f^*(t) dt \geq \int_{I^{n-1}} |f(x)| dx.$$

□

Theorem 4.12. *Let $n = 2$ and $1 < p < \infty$ or $n \geq 3$ and $1 \leq p < \infty$. Let $X(I^{n-1})$ be an r.i. space and let $X_{\mathcal{R}(X, L^\infty)}(I^{n-1})$ be as in (24). Then, the embedding*

$$(25) \quad \mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(X_{\mathcal{R}(X, L^\infty)}, L^{p,1})$$

holds. Moreover, $\mathcal{R}(X_{\mathcal{R}(X, L^\infty)}, L^{p,1})$ is the smallest space of the form $\mathcal{R}(Y, L^{p,1})$ that verifies (25).

Proof. Lemma 4.11 gives us that $X_{\mathcal{R}(X, L^\infty)}(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})}$. Now, let us see that the embedding (25) holds. In fact, if $f \in \mathcal{R}(X, L^\infty)$ then, combining

$$L^\infty(I^n) = \mathcal{R}(L^\infty, L^\infty) \hookrightarrow \mathcal{R}(L^\infty, L^{p,1}),$$

with the embedding due to Algevrik and Kolyada [1], which shows that,

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{p(n-1)/(p(n-1)-1),1}, L^{p,1}),$$

we deduce that

$$K(f, t; \mathcal{R}(L^{p(n-1)/(p(n-1)-1),1}, L^{p,1}), \mathcal{R}(L^\infty, L^{p,1})) \lesssim K(f, Ct; \mathcal{R}(L^1, L^\infty), L^\infty).$$

Hence, using Lemma 3.5 and Theorem 3.6, we get

$$\int_0^t s^{-1/(p(n-1))} \psi_j^*(f, L^{p,1})(s) ds \lesssim \sum_{k=1}^n \int_0^{Ct} \psi_k^*(f, L^\infty)(s) ds, \quad j \in \{1, \dots, n\}.$$

So, using the Hardy-Littlewood inequality and the subadditive of $\|\cdot\|_{\overline{X}(0,|I|^{n-1})}$, we get

$$\begin{aligned} \|f\|_{\mathcal{R}_j(X_{\mathcal{R}(X,L^\infty)}, L^{p,1})} &= \|\psi_j^*(f, L^{p,1})\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} \\ &= \|s^{-1/(p(n-1))}\psi_j^*(f, L^{p,1})(s)\|_{\overline{X}(0,|I|^{n-1})} \\ &\lesssim \sum_{k=1}^n \|\psi_k^*(f, L^\infty)(Ct)\|_{\overline{X}(0,|I|^{n-1})} \lesssim \|f\|_{\mathcal{R}(X,L^\infty)}, \end{aligned}$$

for any $j \in \{1, \dots, n\}$. Therefore, the embedding

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^{p,1})$$

holds, as we wanted to see.

Now, let us prove that $\mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^{p,1})$ is the smallest r.i. space for which (25) holds. That is, let us see that if a mixed norm space $\mathcal{R}(Y, L^{p,1})$ satisfies

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(Y, L^{p,1}),$$

then

$$(26) \quad \mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^{p,1}) \hookrightarrow \mathcal{R}(Y, L^{p,1}).$$

As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any function $f \in X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$, we define

$$g(x) = \begin{cases} |x|^{-1/p} f^*(\omega_{n-1}|x|^{n-1}), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

We fix $k \in \{1, \dots, n\}$. Then,

$$\psi_k(g, L^\infty) = |\widehat{x}_k|^{-1/p} f^*(\omega_{n-1}|\widehat{x}_k|^{n-1}) \quad \text{if } \widehat{x}_k \in B_{n-1}(0, r),$$

and $\psi_k(g, L^\infty)(\widehat{x}_k) = 0$ otherwise. Thus,

$$\|g\|_{\mathcal{R}_k(X,L^\infty)} = \|\psi_k(g, L^\infty)\|_{X(I^{n-1})} \approx \|t^{-1/p(n-1)} f^*(t)\|_{\overline{X}(0,|I|^{n-1})}, \quad k \in \{1, \dots, n\}.$$

Hence, since $f \in X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$, we have that $g \in \mathcal{R}(X, L^\infty)$ and

$$(27) \quad \|g\|_{\mathcal{R}(X,L^\infty)} \approx \|t^{-1/p(n-1)} f^*(t)\|_{\overline{X}(0,|I|^{n-1})}.$$

So, using $\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(Y, L^{p,1})$ and (27), we get

$$(28) \quad \|g\|_{\mathcal{R}(Y,L^{p,1})} \lesssim \|t^{-1/p(n-1)} f^*(t)\|_{\overline{X}(0,|I|^{n-1})} = \|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.$$

Now, let us find a lower estimate for $\|g\|_{\mathcal{R}(Y,L^{p,1})}$. In fact, we fix any $k \in \{1, \dots, n\}$ and $\widehat{x}_k \in B_{n-1}(0, r/2)$. Then, by Hölder's inequality, we get

$$(29) \quad 2^{-1/p'} |\widehat{x}_k|^{-1/p'} \int_{B_1(0, |\widehat{x}_k|)} g(\widehat{x}_k, x_k) dx_k \leq \psi_k(g, L^{p,1})(\widehat{x}_k).$$

Then, by a change of variables, we obtain

$$\begin{aligned} \int_{B_1(0, |\widehat{x}_k|)} g(\widehat{x}_k, x_k) dx_k &\approx \int_{\omega_{n-1} |\widehat{x}_k|^{n-1}}^{2^{n-1} \omega_{n-1} |\widehat{x}_k|^{n-1}} f^*(t) t^{1/(p'(n-1))-1} dt \\ &\geq f^*(2^{n-1} \omega_{n-1} |\widehat{x}_k|^{n-1}) \int_{\omega_{n-1} |\widehat{x}_k|^{n-1}}^{2^{n-1} \omega_{n-1} |\widehat{x}_k|^{n-1}} t^{1/(p'(n-1))-1} dt \\ &\approx |\widehat{x}_k|^{1/p'} f^*(2^{n-1} \omega_{n-1} |\widehat{x}_k|^{n-1}), \end{aligned}$$

and so, using (29), we get

$$f^*(2^{n-1} \omega_{n-1} |\widehat{x}_k|^{n-1}) \lesssim \psi_k(g, L^{p,1})(\widehat{x}_k), \quad \widehat{x}_k \in B_{n-1}(0, r/2).$$

As a consequence, we have

$$\begin{aligned} \|f^*\|_{\overline{Y}(0, |I|^{n-1})} &\lesssim \|f^*(2t) \chi_{(0, \lambda_f(0)/2)}\|_{\overline{Y}(0, |I|^{n-1})} \\ (30) \quad &\lesssim \|\psi_k(g, L^{p,1})\|_{Y(I^{n-1})} = \|g\|_{\mathcal{R}_k(Y, L^{p,1})}. \end{aligned}$$

Therefore, using now (28) and (30), we deduce that

$$\|f\|_{Y(I^{n-1})} \lesssim \|f\|_{X_{\mathcal{R}(X, L^\infty)}(I^{n-1})}.$$

From this, we obtain that any $f \in X_{\mathcal{R}(X, L^\infty)}(I^{n-1})$, with $\lambda_f(0) \leq \omega_n r^{n-1}$, belongs to $Y(I^{n-1})$. The general case can be proved as in the proof of Theorem 4.3. Thus, we have

$$X_{\mathcal{R}(X, L^\infty)}(I^{n-1}) \hookrightarrow Y(I^{n-1}),$$

and so, Lemma 4.2 implies that the embedding (26) holds, as we wanted to see. \square

We observe that, in fact, we have seen that $X_{\mathcal{R}(X, L^\infty)}(I^{n-1})$ is continuously embedded into $Y(I^{n-1})$, which is a stronger condition than (26).

To finish this section, we shall present an application of Theorem 4.12. In particular, we shall see that there is no a smallest space of the form $\mathcal{R}(Y, L^{p,1})$ that would render the embedding due to Algevrik and Kolyada [1]

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{p(n-1)/(p(n-1)-1), 1}, L^{p,1}),$$

true.

Corollary 4.13. *Let $n = 2$ and $1 < p < \infty$ or $n \geq 3$ and $1 \leq p < \infty$. Then, the mixed norm space $\mathcal{R}(L^{pp_1(n-1)/(p(n-1)-p_1), q_1}, L^{p,1})$, with $1 < p_1 < \infty$ and $1 \leq q_1 \leq \infty$ or $p_1 = q_1 = 1$, is the smallest space of the form $\mathcal{R}(Y, L^1)$ satisfying*

$$\mathcal{R}(L^{p_1, q_1}, L^\infty) \hookrightarrow \mathcal{R}(L^{pp_1(n-1)/(p(n-1)-p_1), q_1}, L^{p,1}).$$

Proof. It follows from Theorems 4.12. We only need to use $L^{p_1, q_1}(I^{n-1})$ instead of $X_1(I^{n-1})$. \square

5. FOURNIER EMBEDDINGS

Our main goal, in this section, is to study the following embedding

$$(31) \quad \mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

In particular, we are interested in the following problems:

- (i) Given a mixed norm space $\mathcal{R}(X, L^\infty)$, we would like to find the largest r.i. range space $Z(I^n)$ satisfying (31).

- (ii) Now, let us suppose that the range space is given $Z(I^n)$. We would like to find the largest of the form $\mathcal{R}(X, L^\infty)$ for which (31) holds.

The main motivation to consider these questions come from the embedding due to Fournier [13], which shows that, if $n \geq 2$,

$$(32) \quad \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n).$$

Remark 5.1. We observe that by means of Corollary 3.4, it is possible to prove (32) in a slightly different form. In fact, let $f \in \mathcal{R}(L^1, L^\infty)$. Then, writing $\|f\|_{L^{n',1}(I^n)}$ in terms of the distribution function of f (cf. e.g. [15, Proposition 1.4.9]), and using Corollary 3.4, we get

$$\|f\|_{L^{n',1}(I^n)} = n' \int_0^\infty (\lambda_f(s))^{1/n'} ds \leq n' \int_0^\infty \prod_{k=1}^n (\lambda_{\psi_k(f, L^\infty)}(s))^{1/n} ds.$$

So, the geometric-arithmetic mean inequality implies that

$$\begin{aligned} \|f\|_{L^{n',1}(I^n)} &\leq \frac{1}{(n-1)} \sum_{k=1}^n \int_0^\infty \lambda_{\psi_k(f, L^\infty)}(s) ds = \frac{1}{(n-1)} \sum_{k=1}^n \|\psi_k(f, L^\infty)\|_{L^1(I^{n-1})} \\ &\leq \frac{1}{(n-1)} \sum_{k=1}^n \|f\|_{\mathcal{R}_k(L^1, L^\infty)} = \frac{1}{(n-1)} \|f\|_{\mathcal{R}(L^1, L^\infty)}. \end{aligned}$$

That is, $\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n)$.

5.1. Necessary and sufficient conditions. Now, our main purpose is to find necessary and sufficient conditions on $X(I^{n-1})$ and $Z(I^n)$ under which we have the embedding (31).

Theorem 5.2. *Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces. Then, the embedding*

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$$

holds, if and only if,

$$(33) \quad \|f^*(t^{1/n'})\|_{\overline{Z}(0, |I|^n)} \lesssim \|f^*\|_{\overline{X}(0, |I|^{n-1})}, \quad f \in X(I^{n-1}).$$

Proof. Let us first suppose that the embedding

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$$

holds. As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $0 < r < \min(a, b)$. Given any $f \in X(I^{n-1})$, with $\lambda_f(0) \leq \omega_n^{1/n'} r^{n-1}$, we define

$$g(x) = \begin{cases} f^*(\omega_n^{1/n'} |x|^{n-1}), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

Now, we fix any $k \in \{1, \dots, n\}$. Then,

$$\psi_k(g, L^\infty)(\hat{x}_k) = f^*(\omega_n^{1/n'} |\hat{x}_k|^{n-1}), \quad \text{for any } \hat{x}_k \in B_{n-1}(0, r),$$

and $\psi_k(g, L^\infty)(\hat{x}_k) = 0$ otherwise. Thus, using the boundedness of the dilation operator in r.i. spaces, we get

$$\|g\|_{\mathcal{R}_k(X, L^\infty)} = \|\psi_k(g, L^\infty)\|_{X(I^{n-1})} \lesssim \|f^*\|_{\overline{X}(0, |I|^{n-1})}, \quad k \in \{1, \dots, n\},$$

So, our assumption on f shows that $g \in \mathcal{R}(X, L^\infty)$ and

$$(34) \quad \|g\|_{\mathcal{R}(X, L^\infty)} \lesssim \|f^*\|_{\overline{X}(0, |I|^{n-1})}.$$

Thus, using $\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$ and (34), we obtain

$$(35) \quad \|g\|_{Z(I^n)} \lesssim \|f^*\|_{\overline{X}(0, |I|^{n-1})}.$$

But,

$$\|g\|_{Z(I^n)} = \|f^*(t^{1/n'})\|_{\overline{Z}(0, |I|^n)},$$

and hence (35) gives

$$\|f^*(t^{1/n'})\|_{\overline{Z}(0, |I|^n)} \lesssim \|f\|_{\overline{X}(0, |I|^{n-1})}.$$

This proves (33), for any function $f \in X(I^{n-1})$, with $\lambda_f(0) \leq \omega_n^{1/n'} r^{n-1}$. The general case can be proved as in the proof of Theorem 4.3.

Now, let us suppose that (33) holds. We fix any $f \in \mathcal{R}(X, L^\infty)$ and $s \in (0, |I|^n)$. Then, for any $k \in \{1, \dots, n\}$, it holds that

$$\lambda_{\psi_k(f, L^\infty)}\left(\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'})\right) \leq \lambda_{\psi_k(f, L^\infty)}(\psi_k^*(f, L^\infty)(s^{1/n'})) \leq s^{1/n'}.$$

So, we get

$$\prod_{k=1}^n \lambda_{\psi_k(f, L^\infty)}\left(\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'})\right) \leq s^{n-1}.$$

Hence, Corollary 3.4, with t replaced by $\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'})$, implies that

$$\lambda_f\left(\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'})\right) \leq \left(\prod_{k=1}^n \lambda_{\psi_k(f, L^\infty)}\left(\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'})\right)\right)^{1/(n-1)} \leq s,$$

for any $0 < s < |I|^n$. As a consequence,

$$\sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'}) \in \{y \geq 0 : \lambda_f(y) \leq s\}, \quad \text{for any } 0 < s < |I|^n.$$

Therefore, we obtain

$$f^*(s) \leq \sum_{j=1}^n \psi_j^*(f, L^\infty)(s^{1/n'}), \quad \text{for any } 0 < s < |I|^n.$$

Thus, we have

$$\|f\|_{Z(I^n)} \leq \left\| \sum_{k=1}^n \psi_k^*(f, L^\infty)(s^{1/n'}) \right\|_{\overline{Z}(0, |I|^n)} \leq \sum_{k=1}^n \|\psi_k^*(f, L^\infty)(s^{1/n'})\|_{\overline{Z}(0, |I|^n)}.$$

Hence, using (33), we get

$$\|f\|_{Z(I^n)} \leq \sum_{k=1}^n \|\psi_k^*(f, L^\infty)(s^{1/n'})\|_{\overline{Z}(0, |I|^n)} \lesssim \sum_{k=1}^n \|\psi_k(f, L^\infty)\|_{X(I^{n-1})} = \|f\|_{\mathcal{R}(X, L^\infty)}.$$

That is, the embedding $\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$ holds and the proof is complete. \square

5.2. The optimal domain problem. Let $Z(I^n)$ be an r.i. space. Now, we want to find the largest space of the form $\mathcal{R}(X, L^\infty)$ satisfying

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

In order to do this, let us introduce a new space, denoted by $X_{Z, L^\infty}(I^{n-1})$, consisting of those functions $f \in \mathcal{M}(I^{n-1})$ for which the quantity

$$(36) \quad \|f\|_{X_{Z, L^\infty}(I^{n-1})} = \|f^{**}(s^{1/n'})\|_{\overline{Z}(0, |I|^n)}$$

is finite. It is not difficult to verify that $X_{Z, L^\infty}(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{X_{Z, L^\infty}(I^{n-1})}$.

The next lemma gives an equivalent expression for the norm $\|\cdot\|_{X_{Z, L^\infty}(I^{n-1})}$. The proof follows the same ideas of [12, Theorem 4.4], so we do not include it here.

Lemma 5.3. *Let $Z(I^n)$ be an r.i. space, with $\overline{\alpha}_Z < 1/n'$. Then,*

$$\|f\|_{X_{Z, L^\infty}(I^{n-1})} \approx \|f^*(t^{1/n'})\|_{\overline{Z}(0, |I|^n)}, \quad \text{for any } f \in \mathcal{M}(I^{n-1}).$$

Theorem 5.4. *Let $Z(I^n)$ be an r.i. space, with $\alpha_Z < 1/n'$, and let $X_{Z, L^\infty}(I^{n-1})$ be the r.i. space defined in (36). Then, the embedding*

$$(37) \quad \mathcal{R}(X_{Z, L^\infty}, L^\infty) \hookrightarrow Z(I^n)$$

holds. Moreover, $\mathcal{R}(X_{Z, L^\infty}, L^\infty)$ is the largest space of the form $\mathcal{R}(X, L^\infty)$ for which the embedding (37) holds.

Proof. The embedding (37) follows from Theorem 5.2. Thus, to complete the proof, it only remains to see that $\mathcal{R}(X_{Z, L^\infty}, L^\infty)$ is the largest domain space of the form $\mathcal{R}(X, L^\infty)$ corresponding to $Z(I^n)$. In fact, we shall see that if $\mathcal{R}(Y, L^\infty)$ is another mixed norm space such that (37) holds with $\mathcal{R}(X_{Z, L^\infty}, L^\infty)$ replaced by $\mathcal{R}(Y, L^\infty)$, then

$$\mathcal{R}(Y, L^\infty) \hookrightarrow \mathcal{R}(X_{Z, L^\infty}, L^\infty).$$

We fix any $\mathcal{R}(Y, L^\infty)$. Then, Theorem 5.2 ensures us that

$$\|f^*(t^{1/n'})\|_{\overline{Z}(0, |I|^n)} \lesssim \|f\|_{X(I^{n-1})},$$

and so, using Lemma 5.3, we get

$$\|f\|_{X_{Z, L^\infty}(I^{n-1})} \lesssim \|f\|_{X(I^{n-1})}, \quad f \in X(I^{n-1}).$$

That is, $X(I^{n-1}) \hookrightarrow X_{Z, L^\infty}(I^{n-1})$. Hence, using Theorem 4.3, we deduce that

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(X_{Z, L^\infty}, L^\infty).$$

as we wanted to see. □

Let us see an application of Theorem 5.4.

Corollary 5.5. *Let $n' < p_1 < \infty$, and $1 \leq q_1 \leq \infty$. Then, the mixed norm space $\mathcal{R}(L^{p_1/n', q_1}, L^\infty)$ is the largest space of the form $\mathcal{R}(X, L^\infty)$ satisfying*

$$\mathcal{R}(L^{p_1/n', q_1}, L^\infty) \hookrightarrow L^{p_1, q_1}(I^n).$$

Proof. It follows from Theorem 5.4, with $Z(I^n)$ replaced by $L^{p_1, q_1}(I^n)$. □

5.3. The optimal range problem. Let $X(I^{n-1})$ be an r.i. space. We would like to describe the smallest r.i. space $Z(I^n)$ satisfying

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

We begin with a preliminary lemma.

Lemma 5.6. *Let $X(I^{n-1})$ be an r.i. space. Then, the functional defined by*

$$(38) \quad \|f\|_{Z_{\mathcal{R}(X, L^\infty)}(I^n)} = \|f^*(t^{n'})\|_{\overline{X}(0, |I|^{n-1})}, \quad f \in \mathcal{M}_+(I^n),$$

is an r.i. norm.

Proof. It is enough to apply the same technique as in the proof of Lemma 4.11 . \square

Theorem 5.7. *Let $X(I^{n-1})$ be an r.i. space and let $Z_{\mathcal{R}(X, L^\infty)}(I^n)$ be as in (38) Then, the embedding*

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X, L^\infty)}(I^n)$$

holds. Moreover, $Z_{\mathcal{R}(X, L^\infty)}(I^n)$ is the smallest r.i. space that verifies this embedding.

Proof. Lemma 5.6 gives us that $Z_{\mathcal{R}(X, L^\infty)}(I^n)$ is an r.i. space equipped with the norm $\|\cdot\|_{Z_{\mathcal{R}(X, L^\infty)}(I^n)}$. Now, let us see that the embedding

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X, L^\infty)}(I^n)$$

holds. In fact, let f be any function from $\mathcal{R}(X, L^\infty)$. Then, combining the trivial embedding $L^\infty(I^n) \hookrightarrow L^\infty(I^n)$ with the Fournier's embedding

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n', 1}(I^n),$$

we get

$$K(f, t, L^{n', 1}, L^\infty) \leq K(f, t, \mathcal{R}(L^1, L^\infty), L^\infty), \quad 0 < t < |I|^{n-1}.$$

Thus, Theorem 2.2 and Theorem 3.6, with $X(I^{n-1})$ replaced by $L^1(I^{n-1})$, imply that

$$\int_0^t f^*(s^{n'}) ds \lesssim \sum_{k=1}^n \int_0^t \psi_k^*(f, L^\infty)(s) ds, \quad 0 < t < |I|^{n-1}.$$

Therefore, using Hardy's lemma and the subadditive property of $\|\cdot\|_{\overline{X}(0, |I|^{n-1})}$, we get

$$\|f^*(s^{n'})\|_{\overline{X}(0, |I|^{n-1})} \lesssim \sum_{k=1}^n \|\psi_k^*(f, L^\infty)\|_{\overline{X}(0, |I|^{n-1})} = \|f\|_{\mathcal{R}(X, L^\infty)}.$$

That is, $\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X, L^\infty)}(I^n)$.

Now, let us see that $Z_{\mathcal{R}(X, L^\infty)}(I^n)$ is the smallest r.i. space for which this embedding holds, i.e., let us see that if an r.i. space $Z(I^n)$ satisfies

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n),$$

then $Z_{\mathcal{R}(X, L^\infty)}(I^n) \hookrightarrow Z(I^n)$. As before, assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $0 < r < \min(a, b)$. Given any function $f \in Z_{\mathcal{R}(X, L^\infty)}(I^n)$, with $\lambda_f(0) \leq \omega_n r^n$, we define

$$g(x) = \begin{cases} f^*(\omega_n |x|^n), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

Then, applying the same technique as in the proof of Theorem 4.3 and using the boundedness of the dilation operator in r.i. spaces, we get

$$(39) \quad \|g\|_{\mathcal{R}(X, L^\infty)} \lesssim \|f^*(t^{n'})\|_{\overline{X}(0, |I|^{n-1})} = \|f\|_{Z_{\mathcal{R}(X, L^\infty)}(I^n)}.$$

By hypothesis $f \in Z_{\mathcal{R}(X, L^\infty)}(I^n)$, and hence $g \in \mathcal{R}(X, L^\infty)$. So, using the embedding $\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$ and (39) we get

$$\|g\|_{Z(I^n)} \lesssim \|f\|_{Z_{\mathcal{R}(X, L^\infty)}(I^n)}.$$

But, g and f are equimeasurable functions, and hence we obtain

$$\|f\|_{Z(I^n)} \lesssim \|f\|_{Z_{\mathcal{R}(X, L^\infty)}(I^n)}.$$

From this, we obtain that any $f \in Z_{\mathcal{R}(X, L^\infty)}(I^n)$, with $\lambda_f(0) \leq \omega_n r^n$, belongs to $Z(I^n)$. The general case can be proved as in the proof of Theorem 4.3. Thus, the proof is complete. \square

We shall give now a corollary of Theorems 5.7. In particular, we shall see that the Fournier's embedding (32) cannot be improved within the class of r.i. spaces. This should be understood as follows: if we replace the range space in

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n', 1}(I^n),$$

by a smaller r.i. space, say $Y(I^n)$, then the resulting embedding

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow Y(I^n).$$

can no longer be true.

Corollary 5.8. *Let $1 < p_1 < \infty$ and $1 \leq q_1 \leq \infty$ or $p_1 = q_1 = 1$. Then, the Lorentz space $L^{n'p_1, q_1}(I^n)$ is the smallest r.i. space satisfying*

$$\mathcal{R}(L^{p_1, q_1}, L^\infty) \hookrightarrow L^{n'p_1, q_1}(I^n).$$

Proof. It follows from Theorems 5.7. We only need to use $L^{p_1, q_1}(I^{n-1})$ instead of $X(I^{n-1})$. \square

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